

“To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty of nature. If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in”. -Richard Feynman (1918-1988)

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## Multiplication of Partitioned Matrices

It is sometimes useful to break a large matrix down into smaller submatrices by slicing it up with horizontal or vertical lines that go all the way through the matrix. For example, we can think of the  $4 \times 4$  matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix}$$

as a  $2 \times 2$  **partitioned matrix** whose “entries” are four  $2 \times 2$  matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with  $A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ,  $A_{21} = \begin{bmatrix} 9 & 8 \\ 5 & 4 \end{bmatrix}$ , etc. There is no need for the submatrices to be square or of the same size. For example

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

One of the most useful properties of partitioned matrices is how they behave with respect to matrix multiplication. The following theorem describing how to multiply partitioned matrices is given without proof.

**Theorem** (Partitioned Matrices, Matrix Multiplication - PMMM): Suppose we are given matrices  $A$  and  $B$  for which the product  $AB$  makes sense and which are partitioned into smaller matrices  $A_{ij}$  and  $B_{rs}$  as shown below. Provided all the matrix products  $A_{ik}B_{kj}$  are defined then

$$AB = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rn} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2s} & \cdots & B_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{ns} & \cdots & B_{np} \end{bmatrix}$$

is the partitioned matrix whose  $rs$  th “entry” is the matrix

$$\begin{aligned} [AB]_{rs} &= A_{r1}B_{1s} + A_{r2}B_{2s} + \cdots + A_{rn}B_{ns} \\ &= \sum_{k=1}^n A_{rk}B_{ks}. \end{aligned}$$